Introduction to Mathematical Quantum Theory

Text of the Exercises

– 28.04.2020 –

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Exercise 1

a Let $\mathcal{H}$ be an Hilbert space. Suppose $A, B \in \mathcal{B}(\mathcal{H})$ with $[A, B] = 0$ and A not invertible. Prove that AB is not invertible.

Hint: Prove first that if AB were invertible then A would have both a left and a right inverse. Then prove that those would need to be equal and conclude.

b Prove that if we do not assume A and B to commute, the result in **a** is false.

Exercise 2

Let $\mathcal{H}$ be an Hilbert space. Let A be an unbounded linear operator on $\mathcal{H}$. Suppose there exists a closed operator C that extends the operator A. Prove that A is closable.

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Let A be self-adjoint.

a Suppose $\lambda_0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Prove that

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| = \frac{1}{d(\lambda_0, \sigma(A))}, \tag{1}$$

where $d(x, Y) := \inf_{y \in Y} |x - y|$, with $x \in \mathbb{C}$, $Y \subseteq C$.

Hint: Think of $(A - \lambda_0 \,\mathrm{id})^{-1}$ as a function of A in the sense of the functional calculus of A.

b Let $\lambda_0 \in \mathbb{C}$ and suppose that there exists $\varepsilon > 0$ and some nonzero $\psi \in \mathcal{H}$ such that

$$\|A\psi - \lambda_0 \psi\| < \varepsilon \|\psi\|. \tag{2}$$

Prove that there exists $\lambda \in \sigma(A)$ such that $|\lambda - \lambda_0| < \varepsilon$.

Exercise 4

Let $\mathcal{H} = L^2(I)$, with $I = [0, 1]$. Consider the operator A with domain $D(A) = C(I)$ and with action

$$A\psi(x) = \psi(0), \qquad \forall \psi \in D(A). \tag{3}$$

Prove that A is not closable.